INFORMATION AND CONTROL 10, 409–418 (1967)

Anharmonic Approximation and Bandlimited Signals

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The problem of approximating a given function (in the mean) on a finite interval by a finite sum of nonharmonic oscillations is solved by Hilbert space methods for the case in which the oscillation frequencies are prespecified. Such a problem arises in the mean square approximation of bandlimited signals, in determining the element amplitudes necessary to approximate a given radiation pattern for a linear antenna array with nonuniform element spacings, and in estimating the contribution of a particular reflector when a bandlimited waveform is reflected from a set of fixed reflectors in space.

It is shown that the general approximation problem can be reduced to that of solving the problem for a complex exponential function containing an arbitrary frequency parameter. An intimate connection is also exhibited between the mean square approximation of a bandlimited signal and the problem of interpolating a given set of data points by a bandlimited function of given bandwidth and minimum energy.

I. INTRODUCTION

Let I be given as the union of at most a finite number of disjoint finite intervals on the real line $-\infty < \omega < \infty$ and let B represent the class of all (complex-valued) functions $G(\omega)$ which are defined, measurable and q uare integrable on I; that is,

$$\int_I |G(\omega)|^2 \, d\omega < \infty \quad \text{for} \quad G(\omega) \in B.$$

We shall consider the basic problem of approximating a given member $G(\omega)$ of B on I by a finite series of the form, $\sum_1^N b_n e^{-it_n\omega}$, where the $\{t_i\}_1^N$ are given distinct real numbers and the $\{b_i\}_1^N$ are constant coefficients to be determined by the approximation criterion. The particular criterion chosen here for the optimal coefficients will be that of minimizing the integral of the squared error on I,

$$\left\| G(\omega) - \sum_1^N b_n e^{-it_n\omega} \right\|^2 \triangleq \int_I \left| G(\omega) - \sum_1^N b_n e^{-it_n\omega} \right|^2 d\omega,$$

so that closeness of approximation is measured by the L_2-norm on I.

Such a problem arises in many contexts; for example, if $f(t)$ is a bandlimited function with frequency spectrum vanishing outside $-2\pi W \leqq \omega \leqq 2\pi W$, then the problem considered by Engelman (1963) of approximating $f(t)$ by a series of the form $\sum_1^N b_n[\sin 2\pi W(t - t_n)]/\pi(t - t_n)$ such that $\int_{-\infty}^{\infty} |f - \sum|^2 \, dt$ is minimized reduces to a problem of the type described when reformulated in the frequency domain via Parseval's theorem. In one recent approach to nonuniformly spaced arrays (Unz, 1966), a desired radiation pattern $F(u)$ is to be approximated by a sum $\sum_{n=1}^N A_n e^{ix_n u}$, where u ranges over a finite interval I and the $\{x_n\}_1^N$ describe the spatial positions of the fixed elements. Since Unz (1966) treated the problem by orthogonal methods, the implicit criterion is that

$$\{A_n\}_1^N \text{ be chosen to minimize } \left\| F(u) - \sum_1^N A_n e^{ix_n u} \right\|.$$

Another application arises in the reflection of a known bandlimited waveform from a finite number of fixed reflectors in space. Then the reflected wave has the form

$$R(t) = \sum_1^N A_n f(t - t_n),$$

and the problem of estimating the individual strengths $\{A_n\}_1^N$ from the received waveform $R(t)$ may be approached by minimizing $\| R(t) - \sum_1^N b_n f(t - t_n) \|$ over the b_n's where the norm is now the L_2-norm on $(-\infty, \infty)$. Parseval's theorem can again be used to reduce this problem to the basic approximation problem of the first paragraph.

In Section II, we introduce an inner product in B and find an explicit form for the optimal coefficients in the approximation and the minimum achievable approximation error. The optimal coefficients for any $G(\omega) \in B$ can be derived from those for the specific function $e^{i\xi\omega}$ defined on I and depending on a parameter ξ. This result, proved in Section III, essentially follows from the fact that the optimal coefficients depend linearly on $G(\omega)$. Applications to the approximation of bandlimited functions are given in Section IV and it is shown that the mean square approximation problem for bandlimited functions is intimately related to the problem of finding a minimum energy bandlimited signal which interpolates a finite set of abscissa-ordinate values. This provides both a generalization and an alternate approach to a recent result of L. Levi (1965).

II. APPROXIMATION THEOREM

Let B be defined as in the Introduction; for $F(\omega)$, $G(\omega)$ in B, define an inner product

$$(F(\omega), G(\omega)) \triangleq \int_I F(\omega)G^*(\omega)\, d\omega, \tag{1}$$

where an asterisk denotes complex conjugate. With this definition B becomes a (complex) Hilbert space with induced norm $\|\cdot\|$ given by

$$\| F(\omega)\| \triangleq (F(\omega), F(\omega))^{1/2} = \left(\int_I |F(\omega)|^2\, d\omega\right)^{1/2}. \tag{2}$$

The problem we wish to consider is as follows:

Problem. Given N distinct real numbers $t_1, t_2 \cdots, t_N (N \geqq 1)$ and $G(\omega) \in B$, determine coefficients $b_1, \cdots, b_N$ such that

$$\left\| G(\omega) - \sum_1^N b_n\, e^{-it_n\omega} \right\|$$

is minimized.

THEOREM 1. *For $G(\omega) \in B$ and fixed $t_1, t_2, \cdots t_N$, the optimal coefficients $\{a_n\}$ are determined by the formula*

$$A = 2\pi\Lambda^{-1}S, \tag{3}$$

where Λ is an $N \times N$ matrix with elements given by $\lambda_{jn} = (e^{-it_n\omega}, e^{-it_j\omega})$, A is the $N \times 1$ column matrix of optimal coefficients, S is the $N \times 1$ column matrix of elements $g(t_1), g(t_2), \cdots, g(t_N)$, and

$$g(t) \triangleq \frac{1}{2\pi}\int_I G(\omega)\, e^{it\omega}\, d\omega. \tag{4}$$

(Note that since I is of finite measure, $G(\omega)$ is absolutely integrable on I and $g(t)$ is well-defined for all real t.]

Proof. The functions $\{e^{-it_n\omega}\}_1^N$ are easily verified to be linearly independent on the set I; i.e. only trivial linear combinations of these functions can vanish identically on I. (See Engelman, 1963, Lemma 3, p. 163.) Let L denote the class of all linear combinations of the functions $\{e^{-it_n\omega}\}_1^N$

defined on I. Then L is a linear manifold of B and is closed (in the metric induced by the norm) since it is finite dimensional (Wilansky, 1964, p. 105, Corollary 1). Thus, by known results of approximation theory (Akhiezer-Glazman, 1961, pp. 8–9), there exists one and only one element of L which minimizes the quantity $\| G(\omega) - \sum_1^N b_n e^{-it_n\omega} \|$; that is, there exist (uniquely) optimal coefficient values $\{a_n\}$ such that

$$\left\| G(\omega) - \sum_1^N a_n e^{-it_n\omega} \right\| = \inf_{b_1,\cdots,b_N} \left\| G(\omega) - \sum_1^N b_n e^{-it_n\omega} \right\|.$$

It also follows from general theory (Akhiezer-Glazman, 1961, p. 10) that the error $G(\omega) - \sum_1^N a_n e^{-it_n\omega}$ involved in the optimal approximation must be orthogonal to every element of the subspace L. Equivalently, the error must be orthogonal to each of the N elements $\{e^{-it_n\omega}\}_1^N$ which together span L. Hence a necessary condition for the optimal coefficients is

$$\left(G(\omega) - \sum_1^N a_n e^{-it_n\omega}, e^{-it_j\omega}\right) = 0 \quad \text{for} \quad j = 1, 2, \cdots, N. \tag{5}$$

Expanding the inner product, we obtain the "normal" equations

$$\begin{aligned} &\sum_{n=1}^N a_n(e^{-it_n\omega}, e^{-it_j\omega}) \\ &\qquad = (G(\omega), e^{-it_j\omega}) = 2\pi g(t_j) \quad \text{for} \quad j = 1, 2, \cdots, N, \end{aligned} \tag{6}$$

where $g(t)$ is given by (4). Then, if we denote $(e^{-it_n\omega}, e^{-it_j\omega})$ by λ_{jn}, Eq. (6) becomes

$$\sum_1^N \lambda_{jn} a_n = 2\pi g(t_j) \quad \text{for} \quad j = 1, 2, \cdots, N, \tag{7}$$

or

$$\Lambda A = 2\pi S, \tag{8}$$

where Λ, A and S are matrices defined in the theorem statement. Since $|\Lambda|$ is the Gram determinant of a set of linearly independent functions, it is positive (Akhiezer-Glazman, 1961, p. 13); thus the inverse matrix Λ^{-1} exists and (8) may be solved to yield

$$A = 2\pi\Lambda^{-1}S \tag{8-A}$$

as asserted. Q.E.D.

If we let δ^2 denote the minimum mean square error in the approximation, then by definition

$$\delta^2 \equiv \left\| G(\omega) - \sum_1^N a_n e^{-it_n\omega} \right\|^2$$
$$= \int_I \left| G(\omega) - \sum_1^N a_n e^{-it_n\omega} \right|^2 d\omega.$$

We note that

$$\sum_1^N \sum_1^N \lambda_{jn} a_n a_j^* = \sum_1^N \sum_1^N a_n a_j^* (e^{-it_n\omega}, e^{-it_j\omega})$$
$$= \left(\sum_1^N a_n e^{-it_n\omega}, \sum_1^N a_j e^{-it_j\omega} \right) \geqq 0,$$

so that Λ is a positive definite matrix. Using this fact and some routine manipulation we arrive at the expression

$$\delta^2 = \int_I | G(\omega)|^2 d\omega - A_T^* \Lambda A, \tag{9}$$

where the subscript T denotes matrix transpose. Equivalently, using Parseval's relation and (8-A),

$$\delta^2 = 2\pi \int_{-\infty}^{\infty} | g(t)|^2 dt - 4\pi^2 S_T^* \Lambda^{-1} S. \tag{9-A}$$

III. A QUADRATURE FORMULA FOR THE OPTIMAL COEFFICIENTS

For fixed I and $t_1, t_2, \cdots, t_N$, we wish to show that the approximation problem of Section II for an arbitrary $G(\omega) \in B$ can be solved in terms of the parametric solution for the particular function $e^{-i\xi\omega}$ defined on I, where ξ is a real-valued parameter. From (8-A), the optimal coefficients $c_k(\xi)$ for $e^{-i\xi\omega}$ are given by

$$c_k(\xi) = \sum_{m=1}^N \mu_{km}(e^{-i\xi\omega}, e^{-it_m\omega}) \quad \text{for} \quad k = 1, 2, \cdots, N, \tag{10}$$

where μ_{km} is the term in the mth column and kth row of the inverse matrix Λ^{-1}.

It will be convenient to define the function $\phi(t)$ as follows:

$$\phi(t) \triangleq \frac{1}{2\pi} \int_I e^{it\omega} d\omega, \tag{11}$$

so that the Fourier transform of $\phi(t)$ denoted by $\Phi(\omega)$, has the property

$$\Phi(\omega) = \int_{-\infty}^{\infty} \phi(t)e^{-it\omega}\, dt = \begin{cases} 1 & \text{for} \quad \omega \in I \\ 0 & \text{for} \quad \omega \notin I. \end{cases} \tag{12}$$

Now, consider an arbitrary function $G(\omega) \in B$; the optimal coefficients $\{a_n\}_1^N$, from (8-A), are

$$a_k = \sum_{m=1}^{N} \mu_{km}(G(\omega), e^{-it_m\omega}) \qquad (k = 1, 2, \cdots, N). \tag{13}$$

The following theorem relates the coefficients in (13) to those in Eq. (10):

THEOREM 2. *The optimal coefficients for* $G(\omega)$ *are given by*

$$a_k = \int_{-\infty}^{\infty} g(\xi)c_k(\xi)\, d\xi \quad \text{for} \quad k = 1, 2, \cdots, N, \tag{14}$$

where

$$g(\xi) = \frac{1}{2\pi}\int_I G(\omega)\, e^{i\xi\omega}\, d\omega. \tag{15}$$

Proof. From (10),

$$\begin{aligned} \int_{-\infty}^{\infty} g(\xi)c_k(\xi)\, d\xi &= \int_{-\infty}^{\infty} g(\xi)\left[\sum_{m=1}^{N} \mu_{km}(e^{-i\xi\omega}, e^{-it_m\omega})\right] d\omega \\ &= \sum_{m=1}^{N} \mu_{km} \int_{-\infty}^{\infty} g(\xi)(e^{-i\xi\omega}, \quad e^{-it_m\omega})\, d\omega. \end{aligned} \tag{16}$$

But,

$$(e^{-i\xi\omega}, e^{it_m\omega}) = \int_I e^{-i\xi\omega}\, e^{it_m\omega}\, d\omega = 2\pi\phi(-\xi + t_m), \tag{17}$$

where $\phi(t)$ is defined by (11).

By the generalized Parseval relation for Fourier transforms (Goldberg, 1961, Theorem 13E p. 48):

$$\begin{aligned} \int_{-\infty}^{\infty} g(\xi)\phi(-\xi + t_m)\, d\xi &= \frac{1}{2\pi}\int_I G(\omega)\Phi(\omega)e^{i\omega t_m}\, d\omega \\ &= \frac{1}{2\pi}\int_I G(\omega)e^{i\omega t_m}\, d\omega = \frac{1}{2\pi}(G(\omega), e^{-i\omega t_m}), \end{aligned} \tag{18}$$

having made use of property (12). Combining (16)–(18), we have

$$\int_{-\infty}^{\infty} g(\xi)c_k(\xi)\,d\xi = \sum_{m=1}^{N} \mu_{km}(G(\omega), e^{-i\omega t_m}) = a_k . \qquad \text{Q.E.D.}$$

Thus in the general approximation problem for functions in the Class B, we may restrict our attention (without loss of generality) to the exponential function $e^{-i\xi\omega}$ defined on I. The optimal coefficients for an arbitrary member of B are then recovered by a quadrature as indicated in (14).

IV. APPLICATION TO BANDLIMITED FUNCTIONS

Let $g(t)$ be a continuous complex-valued function which is measurable and square integrable on $-\infty < t < \infty$ with respect to Lebesgue measure. Further assume that the (L_2) Fourier transform, $G(\omega)$, of $g(t)$ vanishes identically for all real ω outside some fixed linear set I, where I, as before, is the union of a finite number of finite disjoint intervals. We will say briefly that a function satisfying these requirements is bandlimited to the set I.

For a given set I, the simplest function bandlimited to I is the function having Fourier transform equal to 1 on I and zero elsewhere; that is,

$$\phi(t) \equiv \frac{1}{2\pi}\int_I e^{it\omega}\,d\omega. \tag{19}$$

For a given $g(t)$ bandlimited to I, our problem is to approximate $g(t)$ in the mean by a fixed linear combination of translates of $\phi(t)$. More precisely, we have:

Problem. Let $t_1, t_2, \cdots, t_N$ be N given distinct real numbers. For $g(t)$ bandlimited to I and $\phi(t)$ defined by (19), choose coefficients $\{b_n\}_1^N$ such that

$$\left\| g(t) - \sum_1^N b_n\,\phi(t - t_n) \right\|$$

is minimized. Here

$$\| g(t)\| = \left(\int_{-\infty}^{\infty} | g(t)|^2\,dt\right)^{1/2}$$

is the usual L_2-norm on $(-\infty, \infty)$.

Example. For $I = [-2\pi W, 2\pi W]$ with $W > 0$, the problem is that of choosing coefficients $\{b_n\}$ to minimize the quantity

$$\int_{-\infty}^{\infty} \left| g(t) - \sum_1^N b_n \frac{\sin 2\pi W(t - t_n)}{\pi(t - t_n)} \right|^2 dt,$$

for a given $g(t)$ bandlimited to $[-2\pi W, 2\pi W]$. Similarly if $I = [-\omega_0 - \pi W, -\omega_0 + \pi W] \cup [\omega_0 - \pi W, \omega_0 + \pi W]$ with $\omega_0 > \pi W > 0$, we have the case of bandpass approximation; that is, the quantity

$$\int_{-\infty}^{\infty} \left| g(t) - \frac{2}{\pi} \sum_1^N b_n \frac{\sin \pi W(t - t_n)}{(t - t_n)} \cos \omega_0(t - t_n) \right|^2 dt$$

is to be minimized for a given $g(t)$ bandlimited to I. Note that the $\phi(t)$ used in each instance is a function of the set I. These two special cases were treated earlier by Engelman (1963) using different methods. The following theorem gives the requisite criterion.

THEOREM 3. *Let $g(t)$ be bandlimited to I and $\phi(t)$ be defined by* (19). *Then, for fixed distinct real numbers $t_1, t_2, \cdots, t_N$, the optimal approximation $g_A(t) \equiv \sum_1^N a_n\phi(t - t_n)$ exists (uniquely) such that*

$$\inf_{b_1,\cdots,b_N} \left\| g(t) - \sum_1^N b_n\, \phi(t - t_n) \right\| = \left\| g(t) - \sum_1^N a_n\, \phi(t - t_n) \right\|.$$

Furthermore, $g_A(t)$ interpolates $g(t)$ at the points $\{t_j\}$; that is

$$g_A(t_j) = g(t_j) \quad \textit{for} \quad j = 1, 2, \cdots, N. \tag{20}$$

Proof. By Parseval's theoem,

$$\inf_{b_1,\cdots,b_N} \left\| g(t) - \sum_1^N b_n\, \phi(t - t_n) \right\| = \inf_{b_1,\cdots,b_N} \frac{1}{2\pi} \left\| G(\omega) - \sum_1^N b_n\, e^{-it_n\omega} \right\|$$

so that the problem reduces to that of Section II. From (7), the optimal coefficients $\{a_n\}$ are determined by

$$\sum_{n=1}^N a_n(e^{-it_n\omega}, e^{-it_j\omega}) = 2\pi g(t_j) \qquad j = 1, 2, \cdots, N. \tag{21}$$

Noting $(e^{-it_n\omega}, e^{-it_j\omega}) = 2\pi\phi(t_j - t_n)$, Eq. (21) becomes

$$\sum_{n=1}^N a_n\phi(t_j - t_n) = g(t_j) \quad \text{for} \quad j = 1, 2, \cdots, N. \tag{22}$$

Since the optimal approximation is $g_A(t) = \sum_1^N a_n\phi(t - t_n)$, it is clear from (22) that $g_A(t_j) = g(t_j)$ as stated. Q.E.D.

Given $g(t)$ bandlimited to I, we define M as the class of all functions

$f(t)$ bandlimited to I which have the additional interpolation property, $f(t_j) = g(t_j)$ for $j = 1, 2, \cdots, N$. Clearly M is nonempty, since $g(t) \in M$ and we have also just shown $g_A(t)$ to be such a function.

THEOREM 4. *For a given $g(t)$ bandlimited to I, the optimal approximation $g_A(t)$ is that member of M having the smallest energy; that is, if $f(t) \in M$, then*

$$\int_{-\infty}^{\infty} | g_A(t)|^2 \, dt \leqq \int_{-\infty}^{\infty} | f(t)|^2 \, dt.$$

Proof. We note first, from (22), that $g_A(t)$ is also the optimal approximation to any $f(t) \in M$. This observation results from the fact that only the values of $\{f(t_j)\}$ are involved in the determination of the optimal coefficients and these values are invariant for the class M. Now, for $f(t) \in M$,

$$\begin{aligned} 0 \leqq \| f(t) - g_A(t)\|^2 &= \int_{-\infty}^{\infty} | f(t) - g_A(t)|^2 \, dt \\ &= \int_{-\infty}^{\infty} | f(t)|^2 \, dt - \psi \, [g(t_1), \cdots, g(t_N)], \end{aligned} \tag{23}$$

where ψ is a non-negative quantity independent of the particular $f \in M$ under consideration. [This follows from Eq. (9-A) and the positive definite nature of Λ^{-1}.] Thus, from (23), the energy of any $f \in M$, namely $\int_{-\infty}^{\infty} | f(t)|^2 \, dt$, will be minimized whenever $\| f(t) - g_A(t)\|$ is minimized. But this latter quantity is clearly minimized (over the class M) when $f(t) \equiv g_A(t)$, thus proving the theorem. Q.E.D.

COROLLARY. *Given a finite set of distinct data points $t_1, t_2, \cdots, t_N$ with corresponding functional values $y_1, y_2, \cdots, y_N$ and a set I (finite number of finite disjoint intervals), then the function $h(t)$ of minimum energy which is bandlimited to I and which interpolates the data points, i.e., $h(t_j) = y_j$, is given by*

$$h(t) = \sum_{1}^{N} a_n \phi(t - t_n), \tag{24}$$

where

$$\phi(t) = \frac{1}{2\pi} \int_I e^{it\omega} \, d\omega, \tag{25}$$

and where the $\{a_n\}_1^N$ *are determined from the set of linear equations*

$$y_j = \sum_1^N a_n \phi(t_j - t_n) \qquad j = 1, 2, \cdots, N. \quad (26)$$

As we have seen, the determinant of the system in (26) cannot vanish since it is the Gram determinant of the linearly independent system $\{e^{-it_j\omega}\}_1^N$ on the set I. Thus, in principle, Cramer's rule provides an explicit solution for the coefficients in the minimum energy interpolating function (24). The result of Levi (1965) is included as a special case; however, the more general approach exhibits the exact role of the set I in determining the functions used in the interpolation and also shows the connection with the mean square approximation of a signal bandlimited to I.

RECEIVED February 24, 1966

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